

$$\sin \theta + \cos \theta = \frac{1}{\sqrt{5}} \text{ の両辺を2乗して}$$

$$(\sin \theta + \cos \theta)^2 = \frac{1}{5} \text{ より}$$

$$\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta = \frac{1}{5}$$

$$1 + 2 \sin \theta \cos \theta = \frac{1}{5}$$

$$2 \sin \theta \cos \theta = -\frac{4}{5}$$

$$\text{よって } \boxed{\sin \theta \cos \theta = -\frac{2}{5}}$$

$$\begin{aligned} \text{また: } \tan \theta + \frac{1}{\tan \theta} &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \\ &= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta \cos \theta} = \boxed{-\frac{5}{2}} \end{aligned}$$

$$\begin{aligned} \tan^3 \theta + \frac{1}{\tan^3 \theta} &= \frac{\sin^3 \theta}{\cos^3 \theta} + \frac{\cos^3 \theta}{\sin^3 \theta} \\ &= \frac{\sin^6 \theta + \cos^6 \theta}{\sin^3 \theta \cos^3 \theta} \\ &= \frac{(\sin^2 \theta + \cos^2 \theta)(\sin^4 \theta - \sin^2 \theta \cos^2 \theta + \cos^4 \theta)}{\sin^3 \theta \cos^3 \theta} \\ &= \frac{(\sin^2 \theta + \cos^2 \theta)^2 - 3 \sin^2 \theta \cos^2 \theta}{\sin^3 \theta \cos^3 \theta} \\ &= \frac{1^2 - 3 \times \left(-\frac{2}{5}\right)^2}{\left(-\frac{2}{5}\right)^3} \\ &= \frac{1 - \frac{12}{25}}{-\frac{8}{125}} = \frac{\frac{13}{25}}{-\frac{8}{125}} = -\frac{13 \times 125}{25 \times 8} = \boxed{-\frac{65}{8}} \end{aligned}$$