(1) $\begin{cases} x = f(\theta) = 2\cos\theta + \cos 2\theta \\ y = g(\theta) = 2\sin\theta - \sin 2\theta \end{cases}$ より

$f\left(\frac{\pi}{3}\right) = 2\cos\frac{\pi}{3} + \cos\frac{2}{3}\pi = 2\times\frac{1}{2} + \left(-\frac{1}{2}\right) = \boxed{\frac{1}{2}}$

$g\left(\frac{\pi}{3}\right) = 2\sin\frac{\pi}{3} - \sin\frac{2}{3}\pi = 2\times\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = \boxed{\frac{\sqrt{3}}{2}}$

(2) $f'(\theta) = -2\sin\theta - 2\sin 2\theta$

$= -2\sin\theta - 2\cdot 2\sin\theta\cos\theta$

$= -2\sin\theta\,(1 + 2\cos\theta)$ よって $0 \leqq \theta \leqq \frac{\pi}{3}$ のとき $f'(\theta) \leqq 0$ だから

$f(\theta)$ は単調減少となる

$g'(\theta) = 2\cos\theta - 2\cos 2\theta$

$= 2\cos\theta - 2\cdot(2\cos^2\theta - 1)$

$= -2\,(2\cos^2\theta - \cos\theta - 1)$

$= -2\,(2\cos\theta + 1)(\cos\theta - 1)$ よって $0 \leqq \theta \leqq \frac{\pi}{3}$ のとき $g'(\theta) \geqq 0$ だから

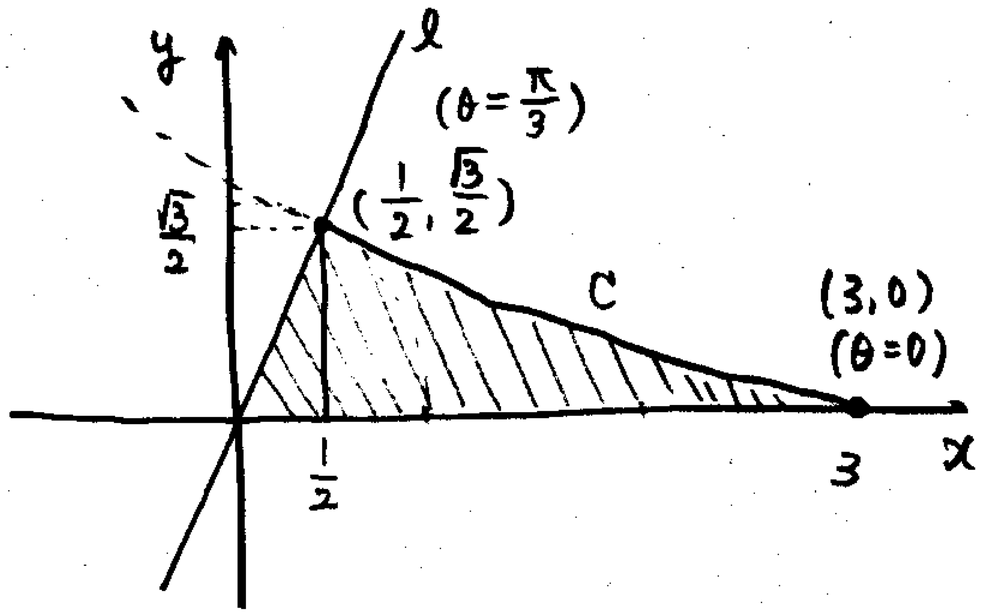
$g(\theta)$ は単調増加となる

(3) $(0,0)$ と $\left(f\left(\frac{\pi}{3}\right), g\left(\frac{\pi}{3}\right)\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ を通る直線 l の方程式は

$\boxed{y = \sqrt{3}x}$

(4) $f(0) = 2 + 1 = 3$, $g(0) = 0$, より $\theta = 0$ のとき $(3,0)$ を通り (2) の結果より

C は左図のようになる



よって求める面積を S とすると

$S = \frac{1}{2}\times\frac{1}{2}\times\frac{\sqrt{3}}{2} + \int_{\frac{1}{2}}^{3} y\,dx$

$S = \frac{\sqrt{3}}{8} + \int_{\frac{\pi}{3}}^{0} g(\theta)\cdot\{-2\sin\theta\,(1 + 2\cos\theta)\}\,d\theta$

$S = \frac{\sqrt{3}}{8} + \int_{\frac{\pi}{3}}^{0} (2\sin\theta - \sin 2\theta)(-2\sin\theta)(2\cos\theta + 1)\,d\theta$

$= \frac{\sqrt{3}}{8} + \int_{\frac{\pi}{3}}^{0} (2\sin\theta - 2\sin\theta\cos\theta)\cdot(-2\sin\theta)(2\cos\theta + 1)\,d\theta$

$= \frac{\sqrt{3}}{8} + \int_{\frac{\pi}{3}}^{0} (-4\sin^2\theta)\cdot(1 - \cos\theta)(2\cos\theta + 1)\,d\theta$

$$= \frac{\sqrt{3}}{8} + 4\int_0^{\frac{\pi}{3}} \sin^2\theta\,(-2\cos^2\theta + \cos\theta + 1)\,d\theta$$

$$= \frac{\sqrt{3}}{8} + 4\int_0^{\frac{\pi}{3}} \left\{-2\cdot\frac{1}{4}(2\sin\theta\cos\theta)^2 + \sin^2\theta\cos\theta + \sin^2\theta\right\} d\theta$$

$$= \frac{\sqrt{3}}{8} + 4\int_0^{\frac{\pi}{3}} \left\{-\frac{1}{2}\cdot\sin^2 2\theta + \sin^2\theta\cos\theta + \sin^2\theta\right\} d\theta$$

$$= \frac{\sqrt{3}}{8} + 4\int_0^{\frac{\pi}{3}} \left\{-\frac{1}{2}\times\frac{1-\cos 4\theta}{2} + \sin^2\theta\cos\theta + \frac{1-\cos 2\theta}{2}\right\} d\theta$$

$$= \frac{\sqrt{3}}{8} + 4\left[-\frac{1}{4}\left(\theta - \frac{1}{4}\sin 4\theta\right) + \frac{1}{3}\sin^3\theta + \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\right]_0^{\frac{\pi}{3}}$$

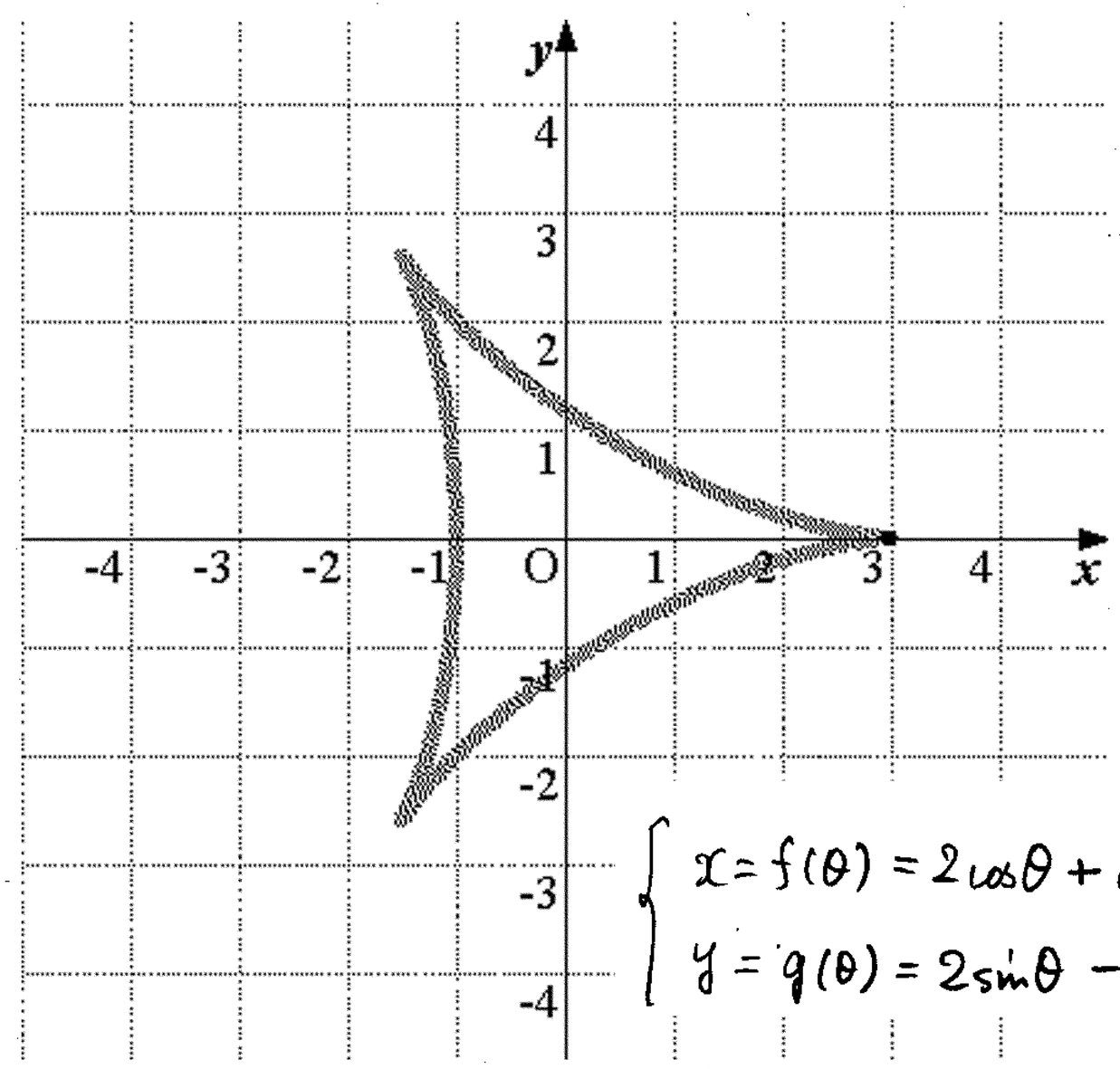
$$= \frac{\sqrt{3}}{8} + 4\left[\frac{1}{4}\theta + \frac{1}{16}\sin 4\theta + \frac{1}{3}\sin^3\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{8} + 4\left\{\frac{1}{4}\times\frac{\pi}{3} + \frac{1}{16}\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3 - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right\}$$

$$= \frac{\sqrt{3}}{8} + 4\left(\frac{\pi}{12} - \frac{\sqrt{3}}{32} + \cancel{\frac{3\sqrt{3}}{24}} - \cancel{\frac{\sqrt{3}}{8}}\right)$$

$$= \frac{\sqrt{3}}{8} + \frac{\pi}{3} - \frac{\sqrt{3}}{8}$$

$$= \boxed{\frac{\pi}{3}}$$



$$\begin{cases} x = f(\theta) = 2\cos\theta + \cos 2\theta \\ y = g(\theta) = 2\sin\theta - \sin 2\theta \end{cases}$$