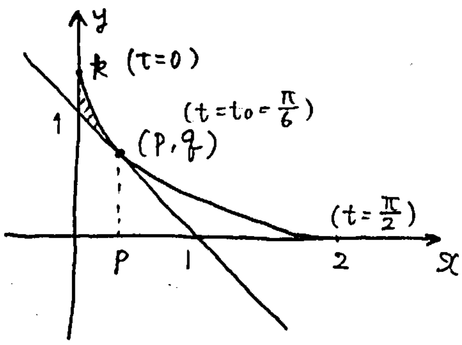
(1) $\begin{cases} x = 2\sin^3 t \\ y = k\cos^3 t \end{cases}$ より $\begin{cases} \dfrac{dx}{dt} = 6\sin^2 t\cos t \\ \dfrac{dy}{dt} = 3k\cos^2 t\,(-\sin t) \end{cases}$ だから

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3\sin t\cos^2 t}{6\sin^2 t\cos t} = \boxed{-\frac{\cos t}{2\sin t}} \quad -①$$

(2) $\sin^2 t = \left(\frac{x}{2}\right)^{\frac{2}{3}}$, $\cos^2 t = \left(\frac{y}{k}\right)^{\frac{2}{3}}$ より $\sin^2 t + \cos^2 t = 1$ から

$\left(\frac{x}{2}\right)^{\frac{2}{3}} + \left(\frac{y}{k}\right)^{\frac{2}{3}} = 1$ をみたすので左図のようになる



Sが第1象限で接するので ① をつかうと

$$\begin{cases} p = 2\sin^3 t_0 & -① \\ q = k\cos^3 t_0 & -② \\ -\dfrac{k\cos t_0}{2\sin t_0} = -1 & -③ \\ p + q = 1 & -④ \end{cases}$$ をみたす

③より $k = 2\tan t_0$ であるから ②は $q = 2\tan t_0\cos^3 t_0$

$= 2\sin t_0\cos^2 t_0$ となる

よって ④は $2\sin^3 t_0 + 2\sin t_0\cos^2 t_0 = 1$ より

$2\sin t_0\,(\sin^2 t_0 + \cos^2 t_0) = 1$ だから

$\sin t_0 = \frac{1}{2}$ より $0 < t_0 < \frac{\pi}{2}$ から $\boxed{t_0 = \frac{\pi}{6}}$

ゆえに $\boxed{k} = 2\tan\frac{\pi}{6} = \frac{2}{\sqrt{3}} = \boxed{\frac{2\sqrt{3}}{3}}$

①より $\boxed{p} = 2\sin^3\frac{\pi}{6} = 2\times\left(\frac{1}{2}\right)^3 = \boxed{\frac{1}{4}}$

$\boxed{q} = k\cos^3\frac{\pi}{6} = \frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{2}\times\left(\frac{\sqrt{3}}{2}\right)^2 = \boxed{\frac{3}{4}}$ となる.

(3) $\displaystyle\int_0^{t_0}\cos^4 t\,dt = \int_0^{t_0}\left(\frac{1+\cos 2t}{2}\right)^2 dt$

$\displaystyle= \frac{1}{4}\int_0^{t_0}(1 + 2\cos 2t + \cos^2 2t)\,dt$

$\displaystyle= \frac{1}{4}\int_0^{t_0}\left(1 + 2\cos 2t + \frac{1+\cos 4t}{2}\right)dt$

$\displaystyle= \frac{1}{4}\int_0^{t_0}\left(\frac{3}{2} + 2\cos 2t + \frac{1}{2}\cos 4t\right)dt$

$\displaystyle= \frac{1}{4}\left[\frac{3}{2} + \sin 2t + \frac{1}{8}\sin 4t\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{4}\times\frac{3}{2}\times\frac{\pi}{6} + \frac{1}{4}\times\sin\frac{\pi}{3} + \frac{1}{32}\times\sin\frac{2}{3}\pi$

$= \frac{3\pi}{48} + \frac{\sqrt{3}}{8} + \frac{1}{32}\times\frac{\sqrt{3}}{2} = \frac{24\pi + 48\sqrt{3} + 6\sqrt{3}}{48\times 8} = \frac{24\pi + 54\sqrt{3}}{384} = \boxed{\frac{4\pi + 9\sqrt{3}}{64}}$

$$\int_0^{t_0} \cos^6 t\,dt = \int_0^{t_0} \cos^5 t\,(\sin t)'\,dt$$

$$= \Big[\cos^5 t \sin t\Big]_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} 5\cos^4 t \sin^2 t\,dt$$

$$= \left(\frac{\sqrt{3}}{2}\right)^5 \times \frac{1}{2} + 5\int_0^{\frac{\pi}{6}} \cos^4 t\,(1-\cos^2 t)\,dt$$

$$= \frac{9\sqrt{3}}{64} + 5\int_0^{\frac{\pi}{6}} \cos^4 t\,dt - 5\int_0^{\frac{\pi}{6}} \cos^6 t\,dt \quad \text{であるから}$$

$$6\int_0^{t_0} \cos^6 t\,dt = \frac{9\sqrt{3}}{64} + 5\int_0^{t_0} \cos^4 t\,dt \quad \text{より}$$

$$\int_0^{t_0} \cos^6 t\,dt = \frac{1}{6}\left(\frac{9\sqrt{3}}{64} + 5 \times \frac{4\pi + 9\sqrt{3}}{64}\right)$$

$$= \frac{20\pi + 54\sqrt{3}}{384}$$

$$= \boxed{\frac{10\pi + 27\sqrt{3}}{192}}$$

(4) 求める面積を S_1 とすると S_1 は (2) の図の斜線部の部分だから

$$S_1 = \int_0^{p} \left\{ k\cos^3 t - (-x+1) \right\} dx \quad \text{であり} \qquad dx = 6\sin^2 t \cos t\,dt, \quad \begin{array}{c|ccc} x & 0 & \cdots & p \\ \hline t & 0 & \cdots & \frac{\pi}{6} \end{array} \quad \text{より}$$

$$S_1 = \int_0^{\frac{\pi}{6}} 6k \sin^2 t \cos^4 t\,dt + \int_0^{p} (x-1)\,dx$$

$$= \cancel{6}^{2} \times \frac{2\sqrt{3}}{\cancel{3}_{1}} \int_0^{\frac{\pi}{6}} (1-\cos^2 t)\cos^4 t\,dt + \left[\frac{1}{2}(x-1)^2\right]_0^{\frac{1}{4}}$$

$$= 4\sqrt{3}\left(\int_0^{t_0} \cos^4 t\,dt - \int_0^{t_0} \cos^6 t\,dt\right) + \frac{1}{2} \times \frac{9}{16} - \frac{1}{2} \times 1$$

$$= 4\sqrt{3} \times \left(\frac{12\pi + 27\sqrt{3}}{192} - \frac{10\pi + 27\sqrt{3}}{192}\right) - \frac{7}{32}$$

$$= 4\sqrt{3} \times \frac{\cancel{2}^{1}\pi}{\cancel{192}_{96}} - \frac{7}{32}$$

$$= \boxed{\frac{4\sqrt{3}\pi - 21}{96}}$$